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Equilibrium in Economy with Linear Production Sets

Agnieszka Lipieta* and Maria Sadko†

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Abstract

Arrow and Debreu proved that, if in the mathematical model of the economy that they defined some conditions are satisfied, then there is a price system under which there is Walras equilibrium in this economy. We examine a procedure, defined on the basis of a continuous mapping, of transformation of production sets of the Arrow-Debreu economies in which at least one of the assumptions concerning the production sets is not satisfied. Finally, we get a class of Arrow-Debreu economies in which, after modification, there is Walras equilibrium. The research methods are based on the analysis of the linear mappings in real vector space of finite dimension, and the results have the form of mathematical theorems.

Keywords: Arrow-Debreu economy, equilibrium, linear sets

JEL Classification: C02, D50

*Krakow University of Economics; e-mail: alipieta@uek.krakow.pl; ORCID: 0000-0002-3017-5755
†Krakow University of Economics; e-mail: maria.sadko@uek.krakow.pl; ORCID: 0000-0003-2736-4982

1 Introduction

In some more recent papers it is shown that linearity of consumption sets (please, see Lipieta 2010, Moore 2007) can, in some cases, lead to equilibrium in Arrow-Debreu economy (Denkowska and Lipieta 2022; Lipieta and Lipieta, 2022, 2023a, 2023b). Maćkowiak (2010) presented a proof of the classical Arrow-Debreu theorem different from this originally printed (1954), whereas Tao (2016) presented some results on long-term equilibrium in Arrow-Debreu economy. In this context, the problem of existence of equilibrium in competitive economy is still explored.
Linearity of consumption sets which is the result of consumers' eco-attitudes may make the producers eliminate harmful commodities or technologies from production (Lipieta and Malawski 2021). However, the incentives for the producers to introduce or adopt eco-innovative solutions may not have their sources in consumers' behaviour. They can be the result of the research, new regulations, taxes, financial penalties for pollution of natural environments, food or water, etc. Therefore, in the paper, we focus on modelling a transformation of a competitive economy that could result in equilibrium in a modification of the economy in which the production sets are linear, without assuming linearity of consumption plans.
Let us recall that Arrow and Debreu in their breakthrough theorem (1954, Theorem 1, p. 272) proved that if in the mathematical model of the economy that they considered (called Arrow-Debreu economy) some conditions are satisfied, then there is a price system under which there is Walras equilibrium in the economy. However, there are some examples of Arrow-Debreu economies in which one, or more than one, among these sufficient conditions are not satisfied (please, for example, see Example 4 in the present paper). Moreover, it can be shown (Example 4), that there is Arrow-Debreu economy for which there is no price system under which there is Walras equilibrium in this economy, as well as at least one of the assumptions of the mentioned classical Arrow-Debreu Theorem (1954) is not satisfied. In the latter case, even linearity of consumption sets would not lead to equilibrium in the system under study.
The above inspired us to examine a feasible transformation of the Arrow-Debreu model in which at least one assumption concerning the properties of production sets from Theorem 1 (i.e. the assumptions I.a, I.b or I.c, Arrow and Debreu 1954, p. 267) is not satisfied and all the rest of the assumptions of this theorem are valid. In the spirit of activity analysis, we aim at defining a continuous mapping that would direct necessary changes to be made into producers' market activities to obtain equilibrium in the economy without changing consumers' market activities. More specifically: we consider Arrow-Debreu economy (Arrow and Debreu 1954) in which assumptions II, III and IV by Theorem 1 (ibidem) are satisfied and aim at transforming the initial economic system to the economy with linear production sets without changing the consumers' characteristics, to attain Arrow-Debreu economy with linear production sets satisfying, additionally the assumptions I.a, I.b and I.c by the Theorem 1 (ibidem). In the final transformation of the initial economy, the

assumptions I – IV will be valid which means that there will be equilibrium in the transformed economy.
Some preliminary results connected with the above issue were presented in (Ulman 2021). Ulman (ibidem) formulated sufficient conditions that guarantee that images under a projection on a linear subspace of the one- or two-dimensional cones with vertex zero, will satisfy conditions I.b or I.c by Theorem 1 in Arrow and Debreu (1954). In the present paper, we significantly extend the results obtained by Ulman (2021). In Proposition 2 we prove, under some assumptions, that there exists equilibrium in the transformation of the initial Arrow-Debreu economy without assuming the linearity of consumption sets. In Example 4, we present Arrow-Debreu economy in which there is no equilibrium, the consumption sets are not linear and, consequently, the procedures of leading the economy to equilibrium presented in (Denkowska and Lipieta 2022) or in (Lipieta and Lipieta 2023a) cannot be applied, while the procedure defined in Proposition 2 leads to equilibrium in a specific transformation of the economy. In Propositions 3 and 4, we provide sufficient conditions for existence of equilibrium in a transformation of the initial economy without assuming linearity of consumption sets. In Remark 2 we present a condition which leads to nonexistence of such a transformation. In Example 5 a mechanism leading to equilibrium in transformations of some Arrow-Debreu economies with one producer and one consumer is presented. To obtain that result, we have to determine the adequate linear dependency between the quantities of some commodities. That enables us to specify a subspace of the commodity-price space that will contain the transformed production plans, as well as a mapping, preferably, linear and continuous which will indicate the trajectory of changes. In the presented research, we use well known features of subspaces of space $\mathbb{R}^\ell$ (see for example Musielak 1989), as well as the properties of projections on linear subspaces of the real space of a finite dimension (see Cheney 1966) to obtain the desired properties of some transformations of the initial economy.
The paper consists of seven parts. The second part presents the literature review, in the third part the reader can find the definition of the model. The fourth part covers the description of the method of transformation of the model, the fifth presents the results. The sixth part is devoted to the discussion, the seventh deals with the conclusions.

2 Literature review

In economic literature, in different contexts, there are analysed issues associated with the problems derived from the general equilibrium theory. The fundamental question that the theorists of economy have been asking themselves was whether there exist any forces which operate on the market and eventually lead it to a state of equilibrium. One of the basic solutions of that problem is the random walk process known also as *tatonnement* (Walras 1874; Arrow, Intriligator 1987). This process is based on the scheme of the market game, in which there exists an auctioneer whose only task is to inform producers and consumers about the current system of prices of goods and services. Being familiar with those prices, producers and consumers are choosing specific action plans, which have to allow them reaching the maximum profits and maximum values of utility functions with the assumption that they are obtainable. If equilibrium will not be reached at given prices, the auctioneer informs about the new system of prices and again he observes the behaviour of entities on the market. This "procedure" lasts until equilibrium prices are determined, as long as the economic system is able to achieve that task in a given situation. Arrow and Debreu (1954) noted that although Walras (1874) was the first to present the concept of equilibrium as a solution of the equations reflecting equilibrium of demand and supply on the market of all commodities, and to formulate the conditions for the existence of equilibrium, he did not include any conclusions showing that such a system has a solution (see also Malawski 1999). In addition, they concluded that under certain conditions imposed on consumers' preferences and producers' feasibilities, the allocation of the goods in a state of general equilibrium is Pareto optimal, which implies that the Pareto optimal allocation is realized in the state of general equilibrium.

Arrow and Hurwicz (1958, 1959) discussed the dynamic stability of the competitive market. In two parts, they proved the existence of stability in a large market where all commodities are substitutes. Debreu (1959) focused on the importance of prices in models in the economy. He explained, among others, how through prices it is possible to interact between entities in the economy with private ownership, and what is the role of prices in the optimal state of the economy. Auman (1964) focused on properties of exchange economy. He analysed two concepts of market equilibrium, namely: competitive equilibrium and equilibrium as a set of equilibrium points.

Marshak (1968) described decentralized pricing mechanisms that can be used by planned economies, companies, and other organizations that are viewed as teams. He considered a good team project, i.e. the one whose members share certain preferences and are expected to respond to the changing environment in a certain desired way. Sonnenschein (1972) studied the class structure of market excess demand functions that can be generated by aggregating individual utility-maximizing behaviours. These considerations resulted in the conclusion that in the relative price domain any polynomial function can be generated as a function of excess demand for a particular good, and that for any p in the relative price domain the configuration of excess demand and the rate of change of excess demand can be generated if and only if Walras' law is satisfied.

Another paper by Sonnenschein (1973), the Walrasian equilibrium model with $n+1$ goods is considered, which generates a class of community excess demand in the domain of relative prices. The main conclusion of the article is that this class generated by Walrasian economies includes all vector-valued function $\left(f^1, \ldots, f^{n+1}\right)$ that are polynomials in the first n coordinates and satisfy Walras identity.

Malawski (1999) used axiomatic method to analyse the problems of general equilibrium. He redefined standard Arrow-Debreu model as a multi-range relational system and put it "in motion" by the use of the concept of a generalized dynamical system.

Mas-Collell, Whinston and Green (1995) presented a methodological approach to an issue, where the economy is considered as a closed system. As a matter of fact, determining the general equilibrium boils down to determining prices and quantities of goods in a system of perfectly competitive markets. This theory is also known as Walrasian theory of markets. Mas-Collell, Whinston and Green (ibidem) introduced issues related to the theory of general equilibrium by three simple examples. They included economies with two consumers, an economy with one consumer and one producer, and a small open economy. Moreover they considered the connection between optimal states. The authors tried to look at the theory of general equilibrium through the prism of game theory. They also showed the application of general equilibrium theory in dynamic modelling of a competitive economy.

Panek (1993) considered the Leontief-Walras equilibrium model and the Walras-Patinkin model, which represent the classical approach to the subject of general equilibrium, yet it has one very serious flaw: in a general case it may turn out that there is no state of equilibrium in it. He presented (ibidem) some theorems from which it follows that, provided the relevant assumptions are met, there is exactly one state of competitive equilibrium in the model under study. Later, Panek (1997) presented dynamic versions of the models considered in (Panek 1993), as well as discussed the stability of the state of equilibrium in a competitive economy.

Magill and Quinzii (2002) studied the effects of trade with the sequential and incomplete market structure. In this situation, a perfect allocation of resources is impossible. Just like in the theory of general equilibrium, the authors made sure that the theory they presented was simple, consistent, general and corresponding to the real situation in the economy.

Maćkowiak (2010) formulated the theory of the existence of a general equilibrium without using the theorem of a fixed point (Brouwer 1911, Kakutani 1939). The assumption adopted in the theorem concerns the correspondence of surplus demand, which is associated with the weak axiom of revealed preference. In addition, the paper included an algorithm for finding equilibrium prices. Tao (2016) made an attempt to formalize the concept of Hayek (1945) about a spontaneous order within the economies of Arrow-Debreu. As a result, he showed that under certain conditions, spontaneous economic order was manifested in a long-term competitive equilibrium, in which participants jointly occupy the optimal distribution of income.

Lipieta (2010) analysed a private ownership economy with complementary commodities. She also examined transformations of the economy being in equilibrium and in which production and consumption sets were linear. Later, Lipieta (2015) analysed also the transformations of Arrow-Debreu economies resulting in equilibrium; however, as the initial model she considered the Arrow-Debreu economy with linear consumptions sets without equilibrium. Lipieta and Malawski (2021) studied the

economic processes that could result in the removal of at least one harmful commodity or harmful technology from the market and presented some examples of such processes which additionally led to equilibrium in the modified economic system. Denkowska and Lipieta (2022), as well as Lipieta and Lipieta (2023b) analysed the possibility of existence of optimal mechanisms in the group of mechanisms leading to equilibrium in a competitive economy.
Due to the best of our knowledge, none of the papers addresses the problem of existence equilibrium in an Arrow-Debreu economy with linear production sets without assuming the linearity of consumption sets. Some preliminary results on this issue were presented by Ulman (2021).

3 The model

We consider the economy defined by Arrow and Debreu (1954) in two periods, $t = 0$ and $t = 1$. We assume that the sets of commodities, the sets of producers, as well as the sets of consumers are the same in both periods. A number of commodities is denoted by $\ell \in \mathbb{N}_+ := \{1, 2, \ldots\}$, a number of consumers by $m \in \mathbb{N}_+$ and a number of producers by $n \in \mathbb{N}_+$. Space $\mathbb{R}^\ell$ with the scalar product

$$(p|x) = ((p_1, \ldots, p_\ell) \,|\, (x_1, \ldots, x_\ell)) = \sum_{k=1}^{\ell} p_k \cdot x_k$$

$(p, x \in \mathbb{R}^\ell)$ is interpreted as a commodity-price space (see Mas-Colell et al. 1995).
Sets $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ stand for (finite) sets of consumers and producers, respectively. A market activity of a producer b_j $(j \in \{1, \ldots, n\})$ at time t is described by an input-output vector $y^{b_j}(t) \in \mathbb{R}^\ell$, called a plan of producer b_j in period t, while an input-output vector $x^{a_i}(t) \in \mathbb{R}^\ell$ $(i \in \{1, \ldots, m\})$, called a plan of consumer a_i in period t, reflects market activity of consumer a_i in period t (for the details, please see, Arrow and Debreu 1954 or Mas-Colell et al. 1995). All production plans with respect to the technologies feasible for producer b_j in period t form the so called production set $Y^{b_j}(t) \subset \mathbb{R}^\ell$. All commodity bundles (consumption plans) that would be wanted to realise by consumer a_i in period t form the consumption set $X^{a_i}(t) \subset \mathbb{R}^\ell$. Under the above presented assumptions and the notion, below an economic system is defined:

Definition 1. *System*

$$\mathcal{E}(t) = \left(\mathbb{R}^\ell, (X^a(t))_{a \in A}, \left(Y^b(t)\right)_{b \in B}, (\preccurlyeq_t^a)_{a \in A}, (\omega^a(t))_{a \in A}, \theta_t, \omega(t)\right),$$

where:

i) $\mathbb{R}^\ell$ *is the commodity and price space,*

ii) $X^a(t) \subset \mathbb{R}^\ell$ *is a non-empty consumption set of consumer* $a \in A$ *at time* t*,* *iii)* $Y^b(t) \subset \mathbb{R}^\ell$ *is a non-empty production set of producer* $b \in B$ *at time* t*,*

iv) $\preccurlyeq_t^a \subset X^a(t) \times X^a(t)$ *– the preference relation (closed, transitive, reflexive and total) of consumer* $a \in A$ *at time* t*,*

v) $\omega^a(t) \in \mathbb{R}^\ell$ *is an initial endowment of consumer* $a \in A$*,*

vi) $\theta_t : A \times B \to [0,1]$ *is a share mapping, where:*

(a) for $a_i \in A$ *and* $b_j \in B$*, number* $\theta_t(a_i, b_j)$*, means the share of consumer* a_i *in the profit of producer* b_j*,*

(b) $\forall b_j \in B \sum_{a_i \in A} \theta_t(a_i, b_j) = 1$*,*

vii) $\omega(t) = \sum_{a_i \in A} \omega^{a_i}(t)$

is called Arrow-Debreu economy (in short: an economy) in period t*.*

Although Definition 1 and the definition presented by Arrow and Debreu (1954) slightly differ, especially in notation, yet they describe the same model of the economy. Economy $\mathcal{E}(1)$ is interpreted as a transformation of economy $\mathcal{E}(0)$. We assume that, in economy $\mathcal{E}(t)$, in every period $t \in \{0,1\}$, the aim of producers is maximizing profits, as well as consumers aim at maximizing preferences on budget sets.
Let $p \in \mathbb{R}^\ell$ mean a vector of commodity prices. For p and $b_j \in B$, we define the set of production plans maximizing the profit of producer b_j under prices p:

$$\eta_t^{b_j}(p) = \left\{ y^{b_j*}(t) \in Y^{b_j}(t) : \left(p|y^{b_j*}(t)\right) = max\left\{ \left(p|y^{b_j}(t)\right) : \ y^{b_j}(t) \in Y^{b_j}(t) \right\} \right\}.$$

Let us notice that set $\eta_t^{b_j}(p)$ can be empty for some $b_j \in B$. Let $p(t)$ mean a vector of commodity prices in period t. According to assumption of perfect rationality (see, for instance, Chapter 7 in Goodwin et al. 2018), it is assumed that, if set $\eta_t^{b_j}(p)$ is not empty, for $p = p(t)$, then, in period t, the producer b_j realizes a production plan $y^{b_j*}(t)$ from the set $\eta_t^{b_j}(p)$. Put

$$\beta_t^{a_i}(p) = \left\{ x^{a_i}(t) \in X^{a_i}(t) : (p|x^{a_i}(t)) \leq (p|\omega^{a_i}(t)) + \sum_{a_i \in A} \theta_t(a_i, b_j) \cdot \left(p|\check{y}^{b_j}(t)\right) \right\},$$

where vector $\check{y}^{b_j}(t)$ describes the market activity of producer b_j in period t. The set $\beta_t^{a_i}(p)$ is the budget set of consumer a_i under prices p. The set $\beta_t^{a_i}(p)$ can be empty. If the set $\beta_t^{a_i}(p)$ is not empty, then we define a set

$$\varphi_t^{a_i}(p) = \left\{ x^{a_i*}(t) \in \beta_t^{a_i}(p) : \ \forall x^{a_i}(t) \in \beta_t^{a_i}(p) \ \ x^{a_i}(t) \preccurlyeq_t^{a_i} x^{a_i*}(t) \right\}.$$

Suppose that, for every $a_i \in A$ and $b_j \in B$ sets $\eta_t^{b_j}(p)$, $\beta_t^{a_i}(p)$ and $\varphi_t^{a_i}(p)$ are not empty. The sequence

$$(x^{a_1*}(t), \ldots, \ x^{a_m*}(t), \ y^{b_1*}(t), \ldots, y^{b_n*}(y), p) \in (\mathbb{R}^\ell)^{m+n+1},$$

where:

(A1) $x^{a_i*}(t) \in \varphi_t^{a_i}(p)$, for every $a_i \in A$ and $y^{b_j*}(t) \in \eta_t^{b_j}(p)$, for every $b_j \in B$,

(A2) $x^{a_1*}(t) + \ldots + x^{a_m*}(t) = y^{b_1*}(t) + \ldots + y^{b_n*}(y) + \omega(t)$,

is called a state of Walras equilibrium (in short: equilibrium) in economy $\mathcal{E}(t)$.
Put $\zeta(t) = x^{a_1*}(t) + \ldots + x^{a_m*}(t) - y^{b_1*}(t) - \ldots - y^{b_n*}(t) - \omega(t)$. It is obvious that, if $\zeta(t) = 0$, then economy $\mathcal{E}(t)$ has equilibrium.

4 Methods

The main tools used in this research for studying existence and some properties of equilibrium states in Arrow-Debreu economy are projections on proper subspaces of space $\mathbb{R}^\ell$. Below we present some definitions and rules useful for further analysis.
Assume that there exists a proper subspace V of space $\mathbb{R}^\ell$ such that

$$\forall a_i \in A \quad X^{a_i}(t) \subset V \tag{1}$$

(see Lipieta 2010, 2015). The sets satisfying condition (1) are called the linear sets. The consequences of the assumption (1) and its economic interpretations were presented and widely analysed, among others, by Lipieta (2010, 2015), Lipieta and Malawski (2021), Denkowska and Lipieta (2022) as well as by Lipieta and Lipieta (2023a). Linear sets were also comprehensively examined by Moore (2007).
Now we recall that, if V is a subspace of dimension $\ell - k$ ($k \in \{1, \ldots, \ell - 1\}$) of space $\mathbb{R}^\ell$, then there exist linearly independent vectors $g^1, \ldots, g^k \in \mathbb{R}^\ell$ ($g^s = (g_1^s, \ldots, g_\ell^s)$, $s \in \{1, 2, \ldots, k\}$) such that

$$V = \bigcap_{s=1}^{k} \ker \widetilde{g}^s, \tag{2}$$

where the mapping

$$\widetilde{g}^s\colon\ \mathbb{R}^\ell \ni (x_1, \ldots, x_\ell) \to g_1^s x_1 + \ldots + g_l^s x_\ell \in \mathbb{R} \tag{3}$$

is, for every $s \in \{1, \ldots, k\}$, linear and continuous, and $\ker \widetilde{g}^s = (\widetilde{g}^s)^{-1}(0)$. Remember that, for a given subspace V, there are infinitely many linearly independent functionals $\widetilde{g}^1, \ldots, \widetilde{g}^k$ of form (3) satisfying condition (2).
Consider vectors $q^1, \ldots, q^k \in \mathbb{R}^\ell$, which are a solution of the system of equations

$$\widetilde{g}^s(q^r) = \delta^{\mathrm{sr}} \text{ for } s, r \in \{1, \ldots, k\}, \tag{4}$$

where

$$\delta^{\mathrm{sr}} = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{if } s \neq r \end{cases}$$

is Kronecker delta. Let a mapping $Q : \mathbb{R}^{\ell} \mapsto \mathbb{R}^{\ell}$ be of the form

$$Q(x) = x - \sum_{s=1}^{k} \widetilde{g}^s(x) \cdot q^s. \tag{5}$$

It is said that vectors $q^1, \ldots, q^k \in \mathbb{R}^{\ell}$ determine mapping Q. Let us notice, that mapping Q is a projection on subspace V, i.e. the following is satisfied:

$$\forall v \in V \ Q(v) = v \text{ and } \forall x \in \mathbb{R}^{\ell} \ Q(x) \in V \tag{6}$$

as well as Q is a linear and continuous mapping (see Cheney 1966). Let us notice that, if $k < \ell$, there are infinitely many solutions of the system of equations (4) and, consequently, infinitely many linear and continuous projections from $\mathbb{R}^{\ell}$ onto subspace V.

Now we present the proposition which gathers the results obtained by Lipieta (2010, 2015) and by Denkowska and Lipieta (2022) on existence of equilibrium in the economy with linear consumption sets. The proposition provides an inspiration for analysing projections in space $\mathbb{R}^{\ell}$ as a possible tool for obtaining equilibrium in a transformation of the economy in which production sets will be linear.

Proposition 1. *Consider an economy $\mathcal{E}(0)$ in which condition (1) is satisfied, as well as there is a sequence*

$$(x^{a_1*}(0), \ldots, \ x^{a_m*}(0), \ y^{b_1*}(0), \ldots, y^{b_n*}(0), p(0)) \in (\mathbb{R}^{\ell})^{m+n+1},$$

for which $(p(0)|\zeta(0)) = 0$, condition (A1) is valid, as well as if $\zeta(0) \neq 0$, then $\zeta(0) \notin V$.

Then there exists a mapping $Q : \mathbb{R}^{\ell} \mapsto V$ of the form (5) such that in economy $\mathcal{E}(1)$ satisfying the following conditions:

(A3) $X^a(0) = X^a(1), \preccurlyeq_0^a = \preccurlyeq_1^a, \omega^a(0) = \omega^a(1)$, *for every $a \in A$, as well as $\theta_1 = \theta_0$,*

(A4) $Y^{b_j}(1) = Q\left(Y^{b_j}(0)\right)$, *for every $b_j \in B$,*

(A5) sequence $\left(x^{a_1*}(1), \ldots, \ x^{a_m*}(1), y^{b_1*}(1), \ldots, y^{b_n*}(1), p(1)\right)$ in which $p(1) = p(0)$, $x^{a_i*}(1) = x^{a_i*}(0)$, for every $a_i \in A$, as well as $y^{b_j*}(1) = Q\left(y^{b_j*}(0)\right)$, for every $b_j \in B$, is a state of Walras equilibrium in economy $\mathcal{E}(1)$.

Proposition 1 says that economy $\mathcal{E}(0)$ with linear consumption sets, in which every producer and every consumer can realize his/her optimal plan of action (i.e. the producers – plan maximizing profits, the consumers – plans maximizing preferences on budget sets), can be transformed into an economy with linear consumption and production sets in which there is a state of equilibrium, if $(p(0)|\zeta(0)) = 0$ and, if $\zeta(0) \neq 0$, then $\zeta(0) \notin V$. Such transformation can be determined by a projection of the form (5). If $(p(0)|\zeta(0)) \neq 0$, some versions of Proposition 1 were proved only under some additional assumptions (see Lipieta 2018).

5 Results

The results of Proposition 1 lead us to the following research question: Does there exist a projection that transforms production sets from economy $\mathcal{E}(0)$ to the sets satisfying assumptions I.a, I.b, I.c from Theorem 1 that was formulated and proved by Arrow and Debreu (1954, p. 267). The existence of such mapping will give equilibrium in economy $\mathcal{E}(1)$ in which conditions II, III and IV considered by Arrow and Debreu (ibidem) are satisfied while the linear production sets are the mentioned transformation of the production sets from economy $\mathcal{E}(0)$.
Let us recall that Arrow and Debreu (1954, p. 267), in their distinguished result (ibidem), consider production sets $Y^{b_1}, \ldots, Y^{b_n} \subset \mathbb{R}^\ell$ (in this research: $Y^{b_j} = Y^{b_j}(0)$ for $j = 1, \ldots, n$) satisfying the following conditions:

I.a. Y^{b_j} is a closed and convex and $0 \in Y^{b_j}$,

I.b. $Y \cap [0, \infty)^l = \{0\}$, for $Y = Y^{b_1} + \ldots + Y^{b_n}$,

I.c. $Y \cap (-Y) = \{0\}$.

Below we briefly summarize some features of projections concerning the above properties.

Remark 1. *Let Z be a subset of $\mathbb{R}^\ell$, V – a subspace of form (2). Consider a projection Q of the form (5) determined by vectors $q^1, \ldots, q^k$ satisfying (4).*
(R1) If Z is a convex set, then set $Q(Z)$ is convex. If set $Q(Z)$ is convex, then the set Z will not have to be convex.
(R2) For the linear subspace

$$W := span\left\{q^1, \ldots, q^k\right\} = \left\{r^1 q^1 + \ldots + r^k q^k : \ r^1, \ \ldots, r^k \in \mathbb{R}\right\} \subset \mathbb{R}^\ell, \tag{7}$$

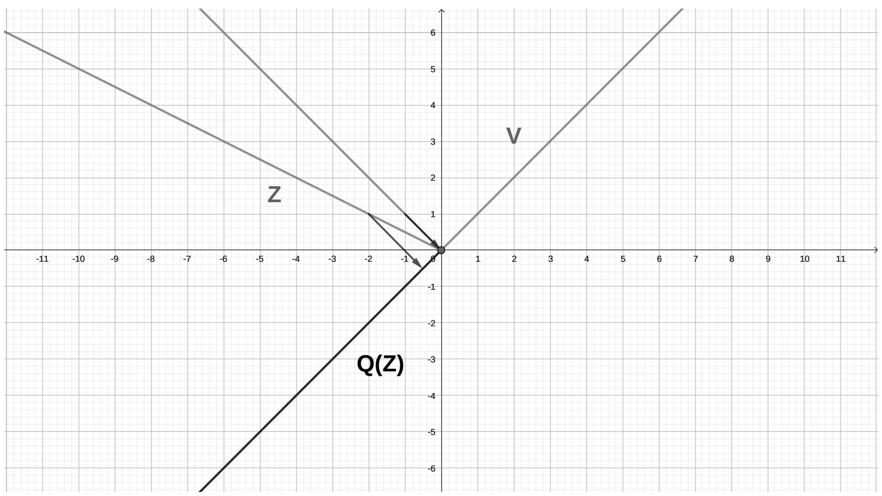
($W = kerQ$) the following is true:

$$Z \cap W \neq \emptyset, \ \textit{if and only, if } 0 \in Q(Z),$$

consequently, if $0 \in Z$, then $0 \in Q(Z)$.
(R3) If Z is closed in $\mathbb{R}^\ell$, then set $Q(Z)$ will not have to be closed in $\mathbb{R}^\ell$.
(R4) If $Z \cap [0, \infty)^\ell = \{0\}$, then the condition I.b will not have to be valid for the set $Q(Z)$.
(R5) If $Z \cap [0, \infty)^\ell \neq \{0\}$, then the condition I.b may be satisfied for the set $Q(Z)$.
(R6) If $Z \cap (-Z) = \{0\}$, then then the condition I.c will not have to be valid for the set $Q(Z)$.
(R7) If $Z \cap (-Z) \neq \{0\}$, then then the condition I.c may be satisfied for the set $Q(Z)$.

Proof. The first statement of part (R1) is the consequence of linearity of projections. Afterwards, we present an example of a nonconvex set Z, a subspace V and a projection Q on space V, which demonstrate the validity of the second statement of part (R1). Put $Z = \{(z_1, z_2) \in \mathbb{R}^2 : (z_2 = -z_1 \vee z_2 = -\frac{1}{2}z_1) \ \wedge \ z_1 \leq 0\}$ and $V = \left\{(v_1,\ v_2) \in \mathbb{R}^2 : v_2 = v_1\right\}$. If Q is of the form (5), where $\widetilde{g}^1\left(x_1,\ x_2\right) = x_1 - x_2$, $q^1 = \left(\frac{1}{2}, -\frac{1}{2}\right)$, (here $k = 1$), we get that $Q(Z) = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = -z_1 \ \wedge \ z_1 \leq 0\}$.

Figure 1: The example of nonconvex set Z and projection Q such that the image of Z under projection Q is convex. Authors own work by the use of GeoGebra



The above means that, if the set $Q(Z)$ is convex, then the set Z will not have to be convex.
The property (R2) is the result of the conditions (5) and (7). The observations (R3)-(R7) are visible in the below examples. $\square$

The example below examines the property (R3).

Example 1. *Consider set* $Z = \left\{(z_1,\ z_2) \in \mathbb{R}^2 :\ z_2 = \frac{-1}{z_1} \wedge z_1 < 0\right\}$. *The set* Z *is closed.*

(1) Put $V = \left\{(v_1,\ v_2) \in \mathbb{R}^2 : v_1 = 0\right\}$ *and* $Q\left(x_1,\ x_2\right) = (0,\ x_2)$. *Then* $Q(Z) = \{0\} \times (0,\ \infty)$ *(please see Figure 2 is not closed (here* $k = 1$, $\widetilde{g}^1\left(x_1,\ x_2\right) = x_1$, $q^1 = (1,\ 0)$*).*

The details can be seen in Figure 2.

(2) If $V = \left\{(v_1,\ v_2) \in \mathbb{R}^2 : v_2 = v_1\right\}$ *and* Q *is of the form (5), where* $\widetilde{g}^1\left(x_1,\ x_2\right) = x_1 - x_2$, $q^1 = \left(\frac{1}{2}, -\frac{1}{2}\right)$ *(here* $k = 1$*), then* $Q(Z) = V$ *(please see Figure 3) and it is closed.*

Figure 2: The example of closed set Z and projection Q such that the image of Z under projection Q is not closed. Authors own work by the use of GeoGebra

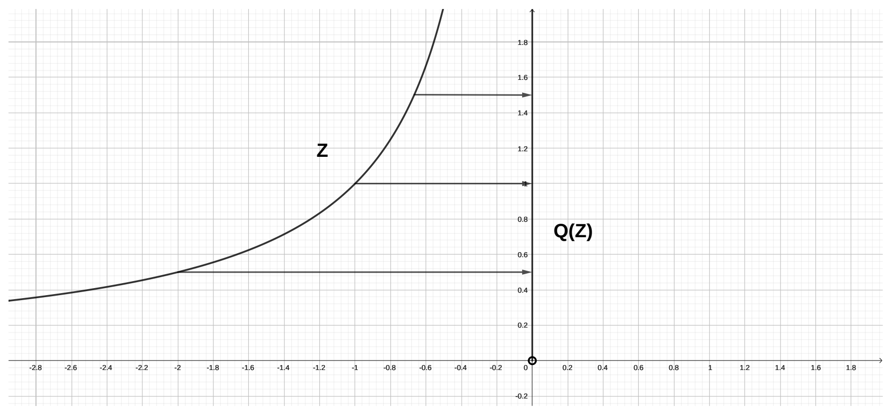
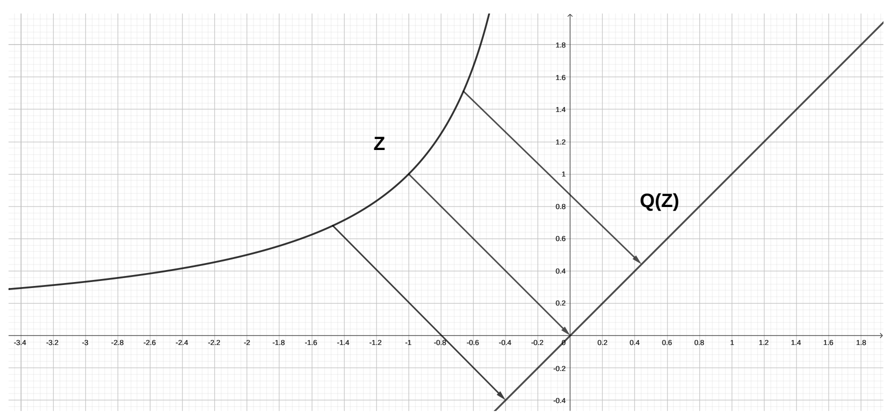


Figure 3: The example of closed set Z and projection Q such that the image of Z under projection Q is closed. Authors own work by the use of GeoGebra



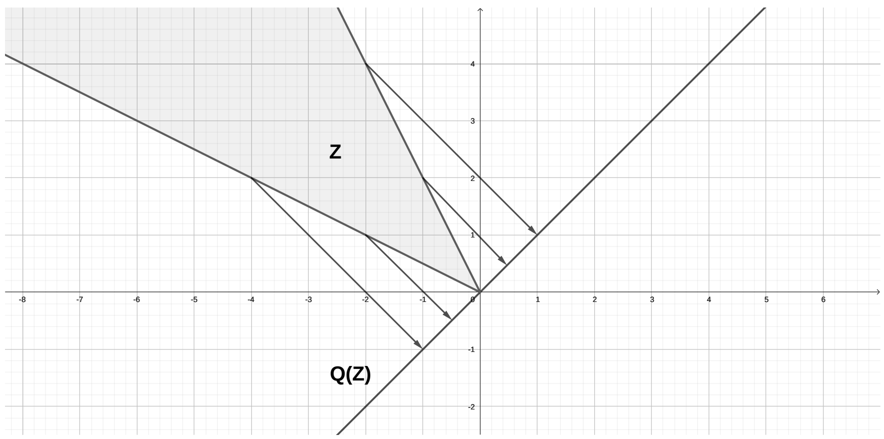
Below the reader can find some examples that substantiate validity of observations (R4) and (R6) by Remark 1.

Example 2. *Consider set* $Z = \left\{(z_1,\ z_2) \in \mathbb{R}^2 :\ z_2 \leq -2z_1 \wedge z_2 \geq -\frac{1}{2}z_1\ \right\}$. *We have* $Z \cap (-Z) = \{0\}$ *and* $Z \cap [0,\infty)^2 = \{0\}$. *Let* $V = \left\{(v_1,\ v_2) \in \mathbb{R}^2 : v_2 = v_1\right\}$, *$Q$ be of the form* (5), *where* $\widetilde{g}^1(x_1,\ x_2) = x_1 - x_2$, $q^1 = \left(\frac{1}{2}, -\frac{1}{2}\right)$ *(here also* $k = 1$*). We can see (Figure 4) that*

(1) $Q(Z) \cap [0,\infty)^2 \neq \{0\}$.

(2) $Q(Z) = V$*,* $Q(-Z) = V$ *and consequently* $Q(Z) \cap Q(-Z) \neq \{0\}$*.*

Figure 4: The example of set Z and projection Q satisfying: $Z \cap (-Z) = \{0\}$, $Z \cap [0, \infty)^2 = \{0\}$, as well as $Q(Z) \cap [0, \infty)^2 \neq \{0\}$ and $Q(Z) \cap Q(-Z) \neq \{0\}$. Authors own work by the use of GeoGebra



In the example below, we present an example of set Z and projection Q for which observation (R7) by Remark 1 is valid, as well as $Z \cap [0, \infty)^\ell = \{0\}$ and $Q(Z) \cap [0, \infty)^2 = \{0\}$.

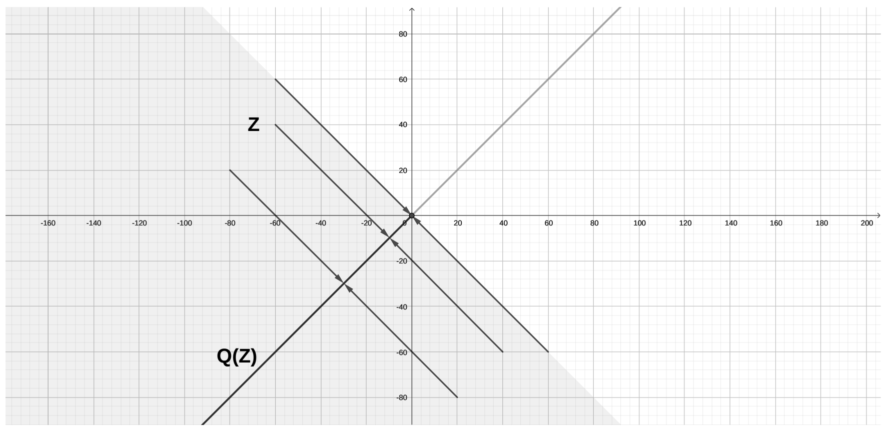
Example 3. *Let* $Z = \{(z_1,\ z_2) \in \mathbb{R}^2 :\ z_2 \leq -z_1\}$*,* $V = \{(v_1,\ v_2) \in \mathbb{R}^2 : v_2 = v_1\}$ *and* Q *is of the form* (5)*, where* $\widetilde{g}^1(x_1,\ x_2) = x_1 - x_2$*,* $q^1 = \left(\frac{1}{2}, -\frac{1}{2}\right)$*. Now* $Q(Z) = \{(v_1,\ v_2) \in \mathbb{R}^2 :\ v_2 = v_1 \wedge v_1 \leq 0\}$*,* $Z \cap (-Z) = \{(z_1,\ z_2) \in \mathbb{R}^2 :\ z_2 = -z_1\}$*. Consequently,* $Z \cap (-Z) \neq \{0\}$*,* $Q(Z) \cap Q(-Z) = \{0\}$*,* $Z \cap [0, \infty)^2 = \{0\}$*,* $Q(Z) \cap [0, \infty)^2 = \{0\}$*.*

Now we present a proposition in which we replace the set of assumptions concerning the production sets of Arrow-Debreu economy $\mathcal{E}(0)$ considered in the Arrow-Debreu Theorem (Arrow, Debreu 1954, p. 272), with the other set of assumptions. The set of assumptions presented below together with the conditions II, III and IV considered in Theorem 1 by Arrow and Debreu (1954, pp. 268-272) give some possibilities to solve the problem of existence of equilibrium in some transformations of the economy $\mathcal{E}(0)$.

Let V be a subspace of form (2) satisfying, additionally, the following condition

$$V \cap [0, \infty)^\ell = \{0\} \tag{8}$$

Figure 5: The example of set Z and projection Q satisfying: $Z \cap (-Z) \neq \{0\}$, $Z \cap [0,\infty)^2 = \{0\}$, as well as $Q(Z) \cap [0,\infty)^2 = \{0\}$ and $Q(Z) \cap Q(-Z) = \{0\}$. Authors own work by the use of GeoGebra



and Q a projection of form (5) determined by vectors $q^1, \ldots, q^k \in \mathbb{R}^\ell$. Let W be of the form (7).

Proposition 2. *Consider private ownership economy* $\mathcal{E}(0)$ *in which:*
(P1a) $\forall b \in B$: *set* $Y^b(0)$ *is convex,*
(P1b) $\forall b \in B$: $Y^b(0) \cap W \neq \emptyset$,
(P1c) $\forall b \in B:\ Q(Y^b(0))$ *is closed,*
(P1d) $y(0) + \check{y}(0) \in W \Rightarrow\ y(0) \in W \wedge \check{y}(0) \in W$,
for every $y(0),\ \check{y}(0) \in Y(0) := Y^{b_1}(0) + \ldots + Y^{b_n}(0)$,
(P2a) $\forall a \in A:\ X^a(0)$ *is closed and convex,*
(P2b) $\forall a \in A\ \exists x \in \mathbb{R}^\ell:\ X^a(0) \subset x + [0,\infty)^\ell$,
(P3a) $\forall a \in A\ \forall x^a(0) \in X^a(0)\ \exists x'^a(0) \in X^a(0)$: $x^a(0) \prec_0^{a_i} x'^a(0)$,
(P3b) $\forall a \in A : \preccurlyeq_0^a$ *satisfies the following:*

$$\forall x^a(0),\ \widetilde{x}^a(0) \in X^a(0):$$

$$((x^a(0) \prec_0^a \widetilde{x}^a(0)) \Longrightarrow (\forall t \in (0,1) : x^a(0) \prec_0^a tx^a(0) + (1-t)\widetilde{x}^a(0))),$$

(P4a) $\forall a \in A\ \exists x^a(0) \in X^a(0)\ \forall l \in \{1,\ldots,\ell\} : x_l^a(0) < \omega_l^a(0)$.
There exists a state of equilibrium in economy $\mathcal{E}(1)$ *in which* $X^a(0) = X^a(1)$, $\preccurlyeq_0^a = \preccurlyeq_1^a$, $\omega^a(0) = \omega^a(1)$, *for every* $a \in A$, *as well as* $\theta_1 = \theta_0$ *and* $Y^{b_j}(1) = Q\left(Y^{b_j}(0)\right)$, *for every* $b_j \in B$.

Proof. If condition (P1a) is satisfied, then set $Q\left(Y^b(0)\right)$ is convex, for every $b \in B$ (please see (R1)). If (P1b) is satisfied, then $0 \in Q\left(Y^b(0)\right)$, for every $b \in B$ (please

see (R2)). Since $V \cap [0,\infty)^\ell = \{0\}$, then $Q(Y(0)) \cap [0,\infty)^\ell = \{0\}$.
Now we show that, if (P1d) is satisfied, then $Q(Y(0)) \cap (-Q(Y(0))) = \{0\}$. By (P1b), $0 \in Q(Y(0)) \cap (-Q(Y(0)))$. If $v \in Q(Y(0)) \cap (-Q(Y(0)))$, then $v = Q(y(0)) = Q(-\check{y}(0))$ for some $y(0),\ \check{y}(0) \in Y(0)$. Hence $Q(y(0) + \check{y}(0)) = 0$. It means that $y(0) + \check{y}(0) \in W$. Since (P1d) is satisfied, then $y(0) \in W$ and $\check{y}(0) \in W$, as well as, consequently, $v = Q(y(0)) = Q(\check{y}(0)) = 0$, which gives that $Q(Y(0)) \cap (-Q(Y(0))) = \{0\}$.
Due to the above, the production sets in economy $\mathcal{E}(1)$, defined in the thesis of the proposition, satisfy assumptions I.a, I.b and I.c from Theorem 1 presented by Arrow and Debreu (1954, pp. 267-272). Assumptions (P2a) and (P2b) give condition II, assumptions (P2a) and (P3b) give condition III.a (see also Mas-Colell et al. 1995, p. 47), assumptions (P2a), (P3a) and (P3b) imply condition III.b, assumptions (P2a) and (P3b) give condition III.c, from the set of the assumptions of the mentioned Theorem 1 (ibidem). The above end the proof of the proposition, because assumption (P4a) is the same as assumption IV.a (ibidem), as well as existence of function θ_0 gives assumption IV.b (ibidem). □

Let us notice that condition $Q(Y(0)) \cap (-Q(Y(0))) = \{0\}$ is also valid, if, among others, dependency $Y(0) + Y(0) \subset W$ is satisfied. Therefore, in Proposition 2, the condition (P1d) can be replaced by this property. Note that, if $Y(0) + Y(0) \subset W$, then total maximal profit in economy $\mathcal{E}(1)$ is equal to 0.
Due to Proposition 2 we also reach the following result: if in economy $\mathcal{E}(0)$ in which every production set is convex, at least one of the assumptions I.a, I.b and I.c of the Theorem 1 (Arrow, Debreu 1954) is not satisfied, then under the assumptions of Proposition 2, it is possible to transform the production sets of economy $\mathcal{E}(0)$ to the sets satisfying these conditions to get equilibrium in economy $\mathcal{E}(1)$ defined in the thesis of Proposition 2. This transformation is determined by continuous and linear mapping Q of the form (5) on a subspace satisfying (8). Moreover, if sets $Y^{b_1}(0), \ldots, Y^{b_n}(0)$ satisfy condition I.a, I.b and I.c (ibidem), then conditions (P1a) and (P1b) are also satisfied in economy $\mathcal{E}(1)$. Due to the above, if, sets $Y^{b_1}(0), \ldots, Y^{b_n}(0)$ satisfies condition I.a, I.b and I.c (ibidem) and, additionally, assumptions (P1c) and (P1d) by Proposition 2 are valid, then it is possible to transform production sets from economy $\mathcal{E}(0)$ in which there exists an equilibrium price vector, on sets contained in a subspace satisfying (8) such that economy $\mathcal{E}(1)$ defined in the thesis of Proposition 2 has an equilibrium price vector.
Let us notice that the procedure presented in Proposition 2 is not determined by the identity mapping, if, for some $b \in B$, $Y^b(0) \cap W \neq \{0\}$ (however, it must be kept $Y^b(0) \cap W \neq \emptyset$, for every $b \in B$). In the above case economies $\mathcal{E}(0)$ and $\mathcal{E}(1)$ differ in their production sets.
Below we present an example of Arrow-Debreu economy with one producer and one consumer that satisfies neither the assumptions of Theorem 1 by Arrow and Debreu (1954) nor the assumptions of Proposition 1. Additionally, we will show that in the given economic system there is no equilibrium, as well as, taking the results

of Proposition 2 into consideration, there is equilibrium in a transformation of the economy.

Example 4. *Let $\mathcal{E}(0)$ be Arrow-Debreu economy in which $A = \{a\}$, $B = \{b\}$, $\ell = 2$, $X^a(0) = [0,\infty) \times [0,4]$, $\omega^a(0) = (4,4)$,*

$$(x_1^a(0), x_2^a(0)) \preccurlyeq_0^a \left(x'^a_1(0), x'^a_2(0)\right) \Leftrightarrow x_1^a(0) + x_2^a(0) \leq x'^a_1(0) + x'^a_2(0),$$

$Y^b(0) = (-\infty,-5] \times [0,\infty)$. It is easy to see that consumption set $X^a(0)$ is not linear. Moreover, we will show:
(I) that in economy $\mathcal{E}(0)$ there is no equilibrium,
(II) how the procedure defined in Proposition 2 works.
***Solution**.*
Ad. I. Let us notice that $\omega^a(0) \notin X^a(0) - Y^b(0) = [5,\infty) \times (-\infty, 4]$. Hence, there is no price vector which will determine equilibrium in economy $\mathcal{E}(0)$. Moreover, in economy $\mathcal{E}(0)$ condition (1) is not satisfied. Therefore we cannot use the procedure described in Proposition 1 to get equilibrium in a transformation of economy $\mathcal{E}(0)$.
Ad. II. It is easy to see that conditions (P2a), (P2b), (P3a) and (P3b) by Proposition 2 are satisfied in economy $\mathcal{E}(0)$. Put

$$V = \left\{(v_1,\ v_2) \in \mathbb{R}^2 : v_2 = -2v_1\right\}, \tag{9}$$

$g(x_1,\ x_2) = 2x_1 +\ x_2$, $q = (\frac{1}{2}, 0)$ and

$$Q(x_1,\ x_2) = (x_1,\ x_2) - (2x_1 +\ x_2) \cdot \left(\frac{1}{2}, 0\right) = \left(-\frac{1}{2}x_2,\ x_2\right) =\ x_2 \cdot (-\frac{1}{2}, 1). \tag{10}$$

It is obvious that $W = \mathbb{R} \times \{0\}$. Subspace V satisfies condition (8) and

$$Q\left(Y^b(0)\right) = \left\{(v_1,\ v_2) \in \mathbb{R}^2 : v_2 = -2v_1 \wedge v_1 \leq 0\right\}.$$

Hence conditions (P1a)-(P1d) by Proposition 2 are satisfied. The latter means that there is equilibrium in economy $\mathcal{E}(1)$ in which $X^a(0) = X^a(1)$, $\preccurlyeq_0^a = \preccurlyeq_1^a$, $\omega^a(0) = \omega^a(1)$, for every $a \in A$, as well as, $\theta_1 = \theta_0$ and $Y^b(1) = Q\left(Y^b(0)\right)$, for Q of form (10). Due to the Proposition 2, in the transformation of economy $\mathcal{E}(0)$ determined by projection Q of form (10), there is a state of Walras equilibrium.

Example 4 shows how the procedure of transformation of initial economy $\mathcal{E}(0)$ presented in Proposition 2 works provided that the assumptions of this proposition are satisfied in the economy $\mathcal{E}(0)$. Let us notice that, in Example 4, the transformed producers' plans satisfy the constant linear dependency between the two commodities. According to the definition presented in (Lipieta 2010) it means that these two commodities are complementary. Hence the transformation defined in Proposition 2 increases the efficiency of the production system.
Proposition 2 inspires to design mechanisms resulting in equilibrium in the transformations of the economy. Below we present a mechanism that transforms

(under the assumptions of Proposition 2) some Arrow-Debreu economies into economies in equilibrium.

Example 5. *Let* V *be the subspace of form* (9) *and* Q *– the projection of form* (10). *To determine a Hurwicz mechanism the following sets and correspondences are specified: set of environments in period* $t = 0$*, set of messages in period* $t = 0$*, a message correspondence, as well as an outcome function (please see, for instance, Hurwicz and Reiter 2006; Denkowska and Lipieta 2022). Consider an Arrow-Debreu economy* $\mathcal{E}(0)$ *(please see Definition 1) with two commodities, one producer* b *and one consumer* a*. Assume that components of the economy* $\mathcal{E}(0)$ *satisfy the assumptions of Proposition 2 with the subspace* V *and the projection* Q*. For every agent* $k \in K = \{a, b\}$ *sequence*

$$e^k(0) = \left(Y^k(0), X^k(0), \omega^k(0), \varepsilon_o(k), \Theta_0(k, \ \cdot)\right), \tag{11}$$

in which:

i) $Y^k(0) = \{0\}$, *if agent* k *is a consumer,*

ii) $X^k(t) = \{0\}$, $\omega^k(t) = 0$, $\varepsilon_0(k) = \{\emptyset\}$, *if agent* k *is a producer,*

iii) mapping $\Theta_0 : K \times K \to [0, 1]$ *is the extension of the mapping* θ_0 *onto set* $K \times K$ *in such a way that* $\Theta_0(a, a) = \Theta_0(b, b) = \Theta_0(b, a) = 0$ *and* $\Theta_0(a, b) = \theta_0(a, b)$

can be assigned. Let, for $k \in K$, $E^k(0)$ *be the set of sequences of the form* (11) *determined by economies with one producer and one consumer. Set* $E(0) \overset{def}{=} E^a(0) \times E^b(0)$ *is the set of feasible economic environments in period* $t = 0$*. It is obvious that the components of the economy* $\mathcal{E}(0)$ *can also be thought as determined by the sequence* $e(0)$*. Under the previous arrangements, the components of every environment* $e(0)$ *are not changed in period* $t = 0$*. Let* $\mathcal{E}(1)$ *be the transformation of an economy* $\mathcal{E}(0)$ *determined by the projection* Q *(see Proposition 2), as well as* T *be the set of equilibrium price vectors in the economy* $\mathcal{E}(1)$*. Due to Proposition 2 the set* T *is not empty. In the economy* $\mathcal{E}(0)$ *the economic agents do not cooperate and do not communicate, hence their market activities are responses to prices on the market. Therefore agents' plans of action can be interpreted as the messages that they send on the market (see Hurwicz 1987). Taking the above into consideration, messages of agent* $k \in K$ *in period* $t = 0$ *are understood as triples* $m^k(0) = (x^k(0), y^k(0), p^k)$*, where*

i) $p^k \in T$,

ii) $x^k(1) \in \varphi_1^k\left(p^k\right)$ *for* $k \in A$; $x^k(0) = 0$ *for* $k \notin A$,

iii) $y^k(0) \in \left\{y^k(0) \in Y^k(0) : \ Q\left(y^k(0)\right) \in \eta_1^b\left(p^k\right)\right\}$ *for* $k \in B$; $y^k(0) = 0$ *for* $k \notin B$.

The set of possible messages of agent k in the environment $e(0)$ *(and consequently, in the economy* $\mathcal{E}(0)$ *determined by the environment* $e(0)$*) is denoted by* $M^k(0)$*. Put*

$$M(0) = \Big\{ \big(x^a(0), y^b(0)\big) : \exists\ m^a = (x^a(0),\ y^a(0), p^a) \in M^a, \\ m^b = \big(x^b(0),\ y^b(0), p^b\big) \in M^b \mid p^a = p^b \Big\}.$$

The sets $M(0)$ *and*

$$Z = \Big\{ (x^a(1), y^b(1)) \in \left(\mathbb{R}^2\right)^2 : \exists\ p \in \mathbb{R}^2 \mid \big(x^a(1), y^b(1), p\big) \\ \textit{is a state of equilibrium in economy } \mathcal{E}(1) \Big\}$$

are not empty due to Proposition 2. If $(x^a(0), y^b(0)) \in M(0)$*, then for* $p = p^a = p^b$*,*

$$y^b(1) = Q(y^b(0)) \in \eta_1^b(p) \wedge x^a(1) = x^a(0) = y^b(1) + \omega^a(0) \in \varphi_1^a(p).$$

Consequently, sequence $\Gamma_0 = (M(0), \mu_0, h_0)$*, where:*

i) $\mu_0 : E(0) \to M(0)$*,* $\mu_0\left(e(0)\right) = M(0)$ *is a message correspondence,*

ii) $h_0 : M(0) \to Z$*,* $h_0\left(m(0)\right) = \Big(Q(y^b(0)) + \omega^a(0), Q(y^b(0))\Big)$*, for* $m(0) \in M(0)$*, is an outcome function,*

is a mechanism in the sense of Hurwicz. The outcomes of the mechanism Γ_0 *are the agents' responses to equilibrium prices. Let us notice that in the mechanism* Γ_0*, in both economies:* $\mathcal{E}(0)$ *and* $\mathcal{E}(1)$*, the total endowments are the same which is coherent with the thesis of Proposition 2.*

An immediate consequence of Proposition 2 is the following proposition.

Proposition 3. *Let* $V \subset \mathbb{R}^\ell$ *be a subspace of dimension* $\ell - k,\ k \in \{1, \ldots,\ \ell - 1\}$ *satisfying* (8) *and* $W \subset \mathbb{R}^\ell$ *– a subspace of dimension* k *such that* $V \cap W = \{0\}$*. If productions sets* $Y^{b_1}(0), \ldots, Y^{b_n}(0)$ *satisfy conditions (P1a), (P1b) and (P1d), then there exists a projection* Q *on subspace* V *such that sets* $Q\left(Y^{b_1}(0)\right), \ldots, Q\left(Y^{b_n}(0)\right)$ *are convex sets containing* 0 *and satisfy the assumptions I.b and I.c by Theorem 1 by Arrow and Debreu (1954).*

Proof. Subspace V is of form (2) for some functionals $\widetilde{g}^s, \ldots, \widetilde{g}^s$. Since $V \cap W = \{0\}$, then there exist linearly independent vectors $q^1, \ldots, q^k \in W$ such that property (4) is satisfied. It is easy to see that the projection Q determined by vectors $q^1, \ldots, q^k$(see (5)) satisfies the thesis of Proposition 3. □

Let us recall that, if sets $Y^{b_1}(0), \ldots, Y^{b_n}(0)$ are compact, then, for every continuous projection Q, sets $Q\left(Y^{b_1}(0)\right), \ldots, Q\left(Y^{b_n}(0)\right)$ are compact and, consequently, closed.

As we can see in Example 1 (please see Figure 2) the image under a projection of a closed but not bounded set do not have to be closed.
To sum up the results obtained, we suggest:

Proposition 4. *Let $V \subset \mathbb{R}^\ell$ be a subspace of dimension $\ell - k,\ k \in \{1, \ldots,\ \ell - 1\}$, satisfying (8) and $W \subset \mathbb{R}^\ell$ – a subspace of dimension k such that $V \cap W = \{0\}$. If compact productions sets $Y^{b_1}(0), \ldots, Y^{b_n}(0)$ satisfy conditions (P1a), (P1b) and (P1d), then there exists a projection Q on subspace V such that the sets $Q\left(Y^{b_1}(0)\right), \ldots, Q\left(Y^{b_n}(0)\right)$ satisfy the assumptions I.a, I.b and I.c by Theorem 1 by Arrow and Debreu (1954).*

Remark 2. *Let us notice that, if $Y^b(0) \subset V \backslash \{0\}$, for some $b \in B$, which means that there is no subspace $W \subset \mathbb{R}^\ell$ of dimension k satisfying both: $V \cap W = \{0\}$ and $Y^b(0) \cap W \neq \emptyset$ (i.e. condition (P1b) is not satisfied, for any subspace $W \subset \mathbb{R}^\ell$ of dimension k such that $V \cap W = \{0\}$), then, for any projection on subspace V, the set $Q(Y^b)$ does not contain zero. The latter means that, in that case, there is no projection on subspace V such that the images of the production sets under that projection would satisfy the assumptions on production sets considered in Theorem 1 by Arrow and Debreu (1954).*

6 Discussion

The procedure presented in Proposition 2 has many implications concerning the properties of transformations of an Arrow-Debreu economy. Let $\mathcal{E}(0)$ be an Arrow-Debreu economy satisfying the assumptions of Proposition 2 and let $\mathcal{E}(1)$ be the economy obtained by the thesis of Proposition 2. If in economy $\mathcal{E}(0)$, for at least one producer $b \in B$, the following condition is satisfied:

$$Q\left(Y^b(0)\right) \subsetneq \bigcup_{b \in B} Y^b(0) \tag{12}$$

which means that in period $t = 1$ there is a production plan that was not feasible in period $t = 0$, then there is an innovation in economy $\mathcal{E}(1)$ with respect to economy $\mathcal{E}(0)$. If in economy $\mathcal{E}(0)$, at least for one producer b,

$$Y^b(0) \subsetneq \bigcup_{b \in B} Q(Y^b(0)), \tag{13}$$

which means that some of the production plans from period $t = 0$ are not feasible in period $t = 1$, then we can see the results of destruction in economy $\mathcal{E}(1)$ with respect to economy $\mathcal{E}(0)$ (please see, Lipieta and Lipieta 2023a). If in economy $\mathcal{E}(0)$ properties (12) and (13) are both satisfied, it means that we note some effects of creative destruction in economy $\mathcal{E}(1)$ with respect to economy $\mathcal{E}(0)$ (ibidem).

Reasoning in the similar way, we get the following conclusion: if in economy $\mathcal{E}(0)$ satisfying the assumptions of Proposition 2, for every producer $b \in B$, all production plans from period $t = 1$ were feasible in period $t = 0$, i.e.,

$$Q\left(Y^b(0)\right) \subset \bigcup_{b \in B} Y^b(0),$$

then there is no innovation in economy $\mathcal{E}(1)$ with respect to economy $\mathcal{E}(0)$.
Linearity of consumption sets which is the result of consumers' convictions or their willingness to get rid of harmful commodities and technologies from the market, can provide the incentives for producers to change their technologies and not to use obsolete and harmful solutions within their activities (please see Denkowska and Lipieta 2022). We believe that Proposition 2 will supply the tools for designing some kind of eco-mechanisms (please see, for instance, Lipieta and Malawski 2021; Denkowska and Lipieta 2022) in the case that was not considered earlier, namely when the consumption sets in economy $\mathcal{E}(0)$ are not linear. However, in the above case there is a necessity to examine methods for modelling the incentives for consumers and producers in order to take part in an eco-mechanism. A similar problem is connected with analysing mechanisms defined in the competitive economy with complementary commodities (Lipieta 2010).
Let us notice that in some Arrow-Debreu models there is no equilibrium at any price vector (see Example 4). Hence a new research problem emerges: in which way the producers could adjust their production sets by adopting new technologies, on the one hand, to maximize profit, on the other hand, to fulfil consumers' requirements. In Arrow-Debreu models, the formulas for the procedures that could result in the above aims can be connected with the geometric properties of the economy under study. The above mentioned procedures would result in an equilibrium in the economy with a reduced production system, provided that all producers would follow the same trajectory of changes, i.e. mapping Q, and the characteristic of consumers would remain the same. Hence, producers' activities should be coordinated and guided except for the case when all producers are motivated for the choice of a given trajectory of changes.
If in period $t = 1$ in the procedure presented in Proposition 2, for at least one producer $b \in B$, there is a production plan that was not feasible in period $t = 0$, then introducing technological innovations is necessary to reach an equilibrium in the transformed system. Introducing innovations requires, among others, investment of a part of producers' profits in the research in new technologies what could lead to the decrease in the quality of life of some market participants. Therefore at the level of the whole economy, an appropriate relationship between the level of investment and the size of consumption expenditures as well as between the size of production and limiting of pollution and using natural resources should be determined. Therefore, designing some mechanisms which will result in eco-innovations in competitive economies and

will not diminish the wealth of economic systems could be the next step of the research presented.

7 Conclusions

This research differs from the traditional papers from the area of general equilibrium. The distinction is that we do not analyse a possibility to reach equilibrium in a considered Arrow-Debreu economy but, in the spirit of activity analysis (see, for example, Koopmans 1951), we examine possibilities of transformation of the production sector of the economy to attain equilibrium in the transformed economy. During such transformations, producers would change their market activities with respect to the initial economy, while the consumers characteristics remain the same.
The above context might be interesting, especially, if there is no equilibrium price vector in the economy. In case of existence of equilibrium in the initial model, the presented procedure also can be applied provided that the assumptions of Proposition 2 are satisfied. It should be added that the transformations of the initial model proposed in the research are defined by the use of a continuous projection on a closed subspace of the commodity space.
The transformed production plans might not be feasible in the initial period. In this case they are interpreted as innovative producers' plans with respect to the initial period. As the set of commodities is not changed during the transformation, then some technologies used in the final period are innovative and introducing technological innovations without changing consumers sets, preferences and endowments are necessary to achieve equilibrium in a transformed system. The transformations that result in innovative changes differ in sets of variables that characterize or will characterize the economic entities, which is caused by adopting by economic agents innovative changes to their routine activities.
Transformed producers plans belong to a subspace of the commodity space. It means that during transformation, producers eliminate some commodities or technologies from their market processes. The possibility of eliminating of a commodity or a technology during a transformation of the economy and modelling the transformation on the basis of a continuous mapping can be interesting for the eco-designers and eco-decision makers who deal with introducing beneficial changes to the environment since determining mechanisms or procedures based on a continuous mapping, assuming *ceteris paribus*, enables to have the modelled changes "under control". Let us notice that not every subspace of the commodity space satisfies the assumption (8), necessary for obtaining the results of Proposition 2, which means that not every procedure or commodity can be eliminated from the production.
If production plans belong to a vector subspace which is determined by functionals in which at least two coordinates are different from zero (please see (2) and (3)), then it means that some commodities used in the production are complementary. Therefore the transformation presented in Proposition 2 makes the production more efficient,

because there is no surplus of commodities in the production plans. The details can be found in (Lipieta 2010).

In case of modification of the production plans, it is important to model a transformation of the economy in such a way that the transformed production sets will be feasible in the first period. In many cases, it would be sufficient only to slightly alter the producers' activity and technologies in such a way that the distances between producers' sets from the final period and production sets from the initial period would be "small". This criterion may be used for defining an optimal transformation (please see Denkowska, Lipieta 2022). To the best of our knowledge, a way of determining of an optimal transformation under the criterion of distance minimization will be probably different than those that have been presented so far. Definition of a sensible criterion of the choice of an optimal transformation among the those defined in Procedure 2 still remains under our research perspective.

In the case of examining mechanisms of a transformation of the economy, some specific incentives for producers shall be taken into account to encourage them to take part in the mechanism and give them space for active participation in modernising obsolete technologies. This is crucial, especially, if consumers do not reveal eco-friendly requirement in their behaviour.

The presented methods do not aspire to the role of the best or most effective and do not solve the problem of nonexistence of equilibrium in Arrow-Debreu economies. They should be rather regarded as an attempt to adapt new concepts for exploring the existing problems for better understanding of the nature of economic equilibrium.

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